

# PROOF OF GROTHENDIECK-SERRE CONJECTURE ON PRINCIPAL BUNDLES OVER REGULAR LOCAL RINGS CONTAINING A FINITE FIELD

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ABSTRACT. Let  $R$  be a regular local ring, containing a **finite field**. Let  $\mathbf{G}$  be a reductive group scheme over  $R$ . We prove that a principal  $\mathbf{G}$ -bundle over  $R$  is trivial, if it is trivial over the fraction field of  $R$ . In other words, if  $K$  is the fraction field of  $R$ , then the map of non-abelian cohomology pointed sets

$$H_{\text{ét}}^1(R, \mathbf{G}) \rightarrow H_{\text{ét}}^1(K, \mathbf{G}),$$

induced by the inclusion of  $R$  into  $K$ , has a trivial kernel.

Certain arguments used in the present preprint do not work if the ring  $R$  contains a characteristic zero field. In that case and, more generally, in the case when the regular local ring  $R$  contains an **infinite field** this result is proved in [FP].

## 1. INTRODUCTION

Assume that  $U$  is a regular scheme,  $\mathbf{G}$  is a reductive  $U$ -group scheme. Recall that a  $U$ -scheme  $\mathcal{G}$  with an action of  $\mathbf{G}$  is called a *principal  $\mathbf{G}$ -bundle over  $U$* , if  $\mathcal{G}$  is faithfully flat and quasi-compact over  $U$  and the action is simple transitive, that is, the natural morphism  $\mathbf{G} \times_U \mathcal{G} \rightarrow \mathcal{G} \times_U \mathcal{G}$  is an isomorphism, see [Gro3, Section 6]. It is well known that such a bundle is trivial locally in étale topology but in general not in Zariski topology. Grothendieck and Serre conjectured that  $\mathcal{G}$  is trivial locally in Zariski topology, if it is trivial generically. More precisely

**Conjecture.** *Let  $R$  be a regular local ring, let  $K$  be its field of fractions. Let  $\mathbf{G}$  be a reductive group scheme over  $U := \text{Spec } R$ , let  $\mathcal{G}$  be a principal  $\mathbf{G}$ -bundle. If  $\mathcal{G}$  is trivial over  $\text{Spec } K$ , then it is trivial. Equivalently, the map of non-abelian cohomology pointed sets*

$$H_{\text{ét}}^1(R, \mathbf{G}) \rightarrow H_{\text{ét}}^1(K, \mathbf{G}),$$

*induced by the inclusion of  $R$  into  $K$ , has a trivial kernel.*

The main result of this paper is a proof of this conjecture for regular semi-local domains  $R$ , containing a **finite field**. Our proof was inspired by the preprint [FP], where the conjecture is proven for semi-local regular domains containing an **infinite field**. Thus, the conjecture holds for semi-local regular domains containing a field.

The proof in the present preprint uses [Pan1, Thm.1.1], [Pan2, Thm.1.0.1], the key ideas of the paper [FP] and a Bertini type theorem from [Poo].

Our result implies that two principal  $\mathbf{G}$ -bundles over  $U$  are isomorphic, if they are isomorphic over  $\text{Spec } K$  as proved in the next section. This result is new even

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for constant group schemes (that is, for group schemes coming from the ground field).

Recall that a part of the Gersten conjecture asserts that the natural homomorphism of  $K$ -groups  $K_q(R) \rightarrow K_q(K)$  is injective. Very roughly speaking, the Grothendieck–Serre conjecture is a non-abelian version of this part of the Gersten conjecture.

**1.1. History of the topic.** Here is a list of known results in the same vein, corroborating the Grothendieck–Serre conjecture.

- The case, where the group scheme  $\mathbf{G}$  comes from an infinite ground field, is completely solved by J.-L. Colliot-Thélène, M. Ojanguren, and M. S. Raghunatan in [CTO] and [Rag1, Rag2]; O. Gabber announced a proof for group schemes coming from arbitrary ground fields.

- The case of an arbitrary reductive group scheme over a discrete valuation ring or over a henselian ring is completely solved by Y. Nisnevich in [Nis1]. He also proved the conjecture for two-dimensional local rings in the case, when  $\mathbf{G}$  is quasi-split in [Nis2].

- The case, where  $\mathbf{G}$  is an arbitrary reductive group scheme over a regular semi-local domain containing an infinite field, was settled by R. Fedorov and I. Panin in [FP].

- The case, where  $\mathbf{G}$  is an arbitrary torus over a regular local ring, was settled by J.-L. Colliot-Thélène and J.-J. Sansuc in [CTS].

- For some simple group schemes of classical series the conjecture is solved in works of the author, A. Suslin, M. Ojanguren, and K. Zainoulline; see [Oja1], [Oja2], [PS1], [OP], [Zai], [OPZ].

- Under an isotropy condition on  $\mathbf{G}$  and assuming that the ring contains an infinite field the conjecture is proved in a series of preprints [PSV] and [Pa2].

- The case of strongly inner simple adjoint group schemes of the types  $E_6$  and  $E_7$  is done by the second author, V. Petrov, and A. Stavrova in [PPS]. No isotropy condition is imposed there, however it is supposed that the ring contains an infinite field.

- The case, when  $\mathbf{G}$  is of the type  $F_4$  with trivial  $g_3$ -invariant and the field is of characteristic zero, is settled by V. Chernousov in [Che]; the case, when  $\mathbf{G}$  is of the type  $F_4$  with trivial  $f_3$ -invariant and the field is infinite and perfect, is settled by V. Petrov and A. Stavrova in [PS2].

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## 2. MAIN RESULTS

Let  $R$  be a commutative unital ring. Recall that an  $R$ -group scheme  $\mathbf{G}$  is called *reductive*, if it is affine and smooth as an  $R$ -scheme and if, moreover, for each algebraically closed field  $\Omega$  and for each ring homomorphism  $R \rightarrow \Omega$  the scalar extension  $\mathbf{G}_\Omega$  is a connected reductive algebraic group over  $\Omega$ . This definition of a reductive  $R$ -group scheme coincides with [DG, Exp. XIX, Definition 2.7]. A well-known conjecture due to J.-P. Serre and A. Grothendieck (see [Ser, Remarque, p.31], [Gro1, Remarque 3, p.26-27], and [Gro2, Remarque 1.11.a]) asserts that

given a regular local ring  $R$  and its field of fractions  $K$  and given a reductive group scheme  $\mathbf{G}$  over  $R$ , the map

$$H_{\text{ét}}^1(R, \mathbf{G}) \rightarrow H_{\text{ét}}^1(K, \mathbf{G}),$$

induced by the inclusion of  $R$  into  $K$ , has a trivial kernel. The following theorem, which is the main result of the present paper, asserts that this conjecture holds, provided that  $R$  contains a **finite field**. If  $R$  contains an infinite field, then the conjecture is proved in [FP].

**Theorem 1.** *Let  $R$  be a regular semi-local domain containing a finite field, and let  $K$  be its field of fractions. Let  $\mathbf{G}$  be a reductive group scheme over  $R$ . Then the map*

$$H_{\text{ét}}^1(R, \mathbf{G}) \rightarrow H_{\text{ét}}^1(K, \mathbf{G}),$$

*induced by the inclusion of  $R$  into  $K$ , has a trivial kernel. In other words, under the above assumptions on  $R$  and  $\mathbf{G}$ , each principal  $\mathbf{G}$ -bundle over  $R$  having a  $K$ -rational point is trivial.*

Theorem 1 has the following

**Corollary.** *Under the hypothesis of Theorem 1, the map*

$$H_{\text{ét}}^1(R, \mathbf{G}) \rightarrow H_{\text{ét}}^1(K, \mathbf{G}),$$

*induced by the inclusion of  $R$  into  $K$ , is injective. Equivalently, if  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are two principal bundles isomorphic over  $\text{Spec } K$ , then they are isomorphic.*

*Proof.* Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two principal  $\mathbf{G}$ -bundles isomorphic over  $\text{Spec } K$ . Let  $\text{Iso}(\mathcal{G}_1, \mathcal{G}_2)$  be the scheme of isomorphisms. This scheme is a principal  $\text{Aut } \mathcal{G}_2$ -bundle. By Theorem 1 it is trivial, and we see that  $\mathcal{G}_1 \cong \mathcal{G}_2$ .  $\square$

Note that, while Theorem 1 was previously known for reductive group schemes  $\mathbf{G}$  coming from the ground field (an unpublished result due to O.Gabber), in many cases the corollary is a new result even for such group schemes.

For a scheme  $U$  we denote by  $\mathbb{A}_U^1$  the affine line over  $U$  and by  $\mathbb{P}_U^1$  the projective line over  $U$ . Let  $T$  be a  $U$ -scheme. By a principal  $\mathbf{G}$ -bundle over  $T$  we understand a principal  $\mathbf{G} \times_U T$ -bundle.

In Section 3 we deduce Theorem 1 from the following result of independent interest (cf. [PSV, Thm.1.3]).

**Theorem 2.** *Let  $R$  be the semi-local ring of finitely many closed points on an irreducible smooth affine variety over a finite field  $k$ , set  $U = \text{Spec } R$ . Let  $\mathbf{G}$  be a simple simply-connected group scheme over  $U$  (see [DG, Exp. XXIV, Sect. 5.3] for the definition). Let  $\mathcal{E}_t$  be a principal  $\mathbf{G}$ -bundle over the affine line  $\mathbb{A}_U^1 = \text{Spec } R[t]$ , and let  $h(t) \in R[t]$  be a monic polynomial. Denote by  $(\mathbb{A}_U^1)_h$  the open subscheme in  $\mathbb{A}_U^1$  given by  $h(t) \neq 0$  and assume that the restriction of  $\mathcal{E}_t$  to  $(\mathbb{A}_U^1)_h$  is a trivial principal  $\mathbf{G}$ -bundle. Then for each section  $s : U \rightarrow \mathbb{A}_U^1$  of the projection  $\mathbb{A}_U^1 \rightarrow U$  the  $\mathbf{G}$ -bundle  $s^* \mathcal{E}_t$  over  $U$  is trivial.*

The derivation of Theorem 1 from Theorem 2 is based on [Pan2, Thm.1.0.1] and [Pan1, Thm.1.1].

Let  $Y$  be a semi-local scheme. We will call a simple  $Y$ -group scheme quasi-split if its restriction to each connected component of  $Y$  contains a **Borel subgroup scheme**.

**Theorem 3.** *Let  $R$ ,  $U$ , and  $\mathbf{G}$  be as in Theorem 2. Let  $Z \subset \mathbb{P}_U^1$  be a closed subscheme finite over  $U$ . Let  $Y \subset \mathbb{P}_U^1$  be a closed subscheme finite and étale over  $U$  and such that*

- (i)  $\mathbf{G}_Y := \mathbf{G} \times_U Y$  is quasi-split,
  - (ii)  $Y \cap Z = \emptyset$  and  $Y \cap \{\infty\} \times U = \emptyset = Z \cap \{\infty\} \times U$ ,
  - (iii) for any closed point  $u \in U$  one has  $\text{Pic}(\mathbb{P}_u^1 - Y_u) = 0$ , where  $Y_u := \mathbb{P}_u^1 \cap Y$ .
- Let  $\mathcal{G}$  be a principal  $\mathbf{G}$ -bundle over  $\mathbb{P}_U^1$  such that its restriction to  $\mathbb{P}_U^1 - Z$  is trivial. Then the restriction of  $\mathcal{G}$  to  $\mathbb{P}_U^1 - Y$  is also trivial. In particular, the principal  $\mathbf{G}$ -bundle  $\mathcal{G}$  is trivial locally for the Zarisky topology.*

The proof of this result is inspired by [FP, Thm.3].

**2.1. Organization of the paper.** In Section 3, we reduce Theorem 1 to Theorem 2. In Section 4, we reduce Theorem 2 to Theorem 3. This reduction is based on [Pan2, Thm.1.0.1], [Pan1, Thm.1.1], on a theorem of D. Popescu [Pop] and on Proposition 4.1. The latter proposition is a new ingredient comparing with respecting arguments from [FP, Section 4].

In Section 5 we prove Theorem 3. We give an outline of the proof in Section 5.1. We use the technique of henselization.

In Section 6 we give an application of Theorem 1.

In the Appendix we recall the definition of henselization from [Gab, Section 0].

### 3. REDUCING THEOREM 1 TO THEOREM 2

In what follows “ $\mathbf{G}$ -bundle” always means “principal  $\mathbf{G}$ -bundle”. Now we assume that Theorem 2 holds. We start with the following particular case of Theorem 1.

**Proposition 3.1.** *Let  $R$ ,  $U = \text{Spec } R$ , and  $\mathbf{G}$  be as in Theorem 2. Let  $\mathcal{E}$  be a principal  $\mathbf{G}$ -bundle over  $U$ , trivial at the generic point of  $U$ . Then  $\mathcal{E}$  is trivial.*

*Proof.* Under the hypothesis of the proposition, the following data are constructed in [Pan1, Thm.1.1]:

- (a) a principal  $\mathbf{G}$ -bundle  $\mathcal{E}_t$  over  $\mathbb{A}_U^1$ ;
- (b) a monic polynomial  $h(t) \in R[t]$ .

Moreover these data satisfies the following conditions:

- (1) the restriction of  $\mathcal{E}_t$  to  $(\mathbb{A}_U^1)_h$  is a trivial principal  $\mathbf{G}$ -bundle;
- (2) there is a section  $s : U \rightarrow \mathbb{A}_U^1$  such that  $s^* \mathcal{E}_t = \mathcal{E}$ .

Now it follows from Theorem 2 that  $\mathcal{E}$  is trivial.  $\square$

**Proposition 3.2.** *Let  $U$  be as in Theorem 2. Let  $\mathbf{G}$  be a reductive group scheme over  $U$ . Let  $\mathcal{E}$  be a principal  $\mathbf{G}$ -bundle over  $U$  trivial at the generic point of  $U$ . Then  $\mathcal{E}$  is trivial.*

*Proof.* Firstly, using [Pan2, Thm.1.0.1], we can assume that  $\mathbf{G}$  is semi-simple and simply-connected. Secondly, standard arguments (see for instance [PSV, Section 9]) show that we can assume that  $\mathbf{G}$  is simple and simply-connected. (Note that for this reduction it is necessary to work with semi-local rings.) Now the proposition is reduced to Proposition 3.1.  $\square$

*Proof of Theorem 1.* Let us prove a general statement first. Let  $k'$  be a **finite field**,  $X$  be a  $k'$ -smooth irreducible affine variety,  $\mathbf{H}$  be a reductive group scheme over  $X$ . Denote by  $k'[X]$  the ring of regular functions on  $X$  and by  $k'(X)$  the field of rational functions on  $X$ . Let  $\mathcal{H}$  be a principal  $\mathbf{H}$ -bundle over  $X$  trivial over  $k'(X)$ .

Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  be prime ideals in  $k'[X]$ , and let  $\mathcal{O}_{\mathfrak{p}_1, \dots, \mathfrak{p}_n}$  be the corresponding semi-local ring.

**Lemma 3.3.** *The principal  $\mathbf{H}$ -bundle  $\mathcal{H}$  is trivial over  $\mathcal{O}_{\mathfrak{p}_1, \dots, \mathfrak{p}_n}$ .*

*Proof.* For each  $i = 1, 2, \dots, n$  choose a maximal ideal  $\mathfrak{m}_i \subset k'[X]$  containing  $\mathfrak{p}_i$ . One has inclusions of  $k'$ -algebras

$$\mathcal{O}_{\mathfrak{m}_1, \dots, \mathfrak{m}_n} \subset \mathcal{O}_{\mathfrak{p}_1, \dots, \mathfrak{p}_n} \subset k'(X).$$

By Proposition 3.2 the principal  $\mathbf{H}$ -bundle  $\mathcal{H}$  is trivial over  $\mathcal{O}_{\mathfrak{m}_1, \dots, \mathfrak{m}_n}$ . Thus it is trivial over  $\mathcal{O}_{\mathfrak{p}_1, \dots, \mathfrak{p}_n}$ .  $\square$

Let us return to our situation. Let  $\mathfrak{m}_1, \dots, \mathfrak{m}_n$  be all the maximal ideals of  $R$ . Let  $\mathcal{E}$  be a  $\mathbf{G}$ -bundle over  $R$  trivial over the fraction field of  $R$ . Clearly, there is a non-zero  $f \in R$  such that  $\mathcal{E}$  is trivial over  $R_f$ . Let  $k$  be the prime field of  $R$ . Note that  $k$  is perfect. It follows from Popescu's theorem ([Pop, Swa]) that  $R$  is a filtered inductive limit of smooth  $k$ -algebras  $R_\alpha$ . Modifying the inductive system  $R_\alpha$  if necessary, we can assume that each  $R_\alpha$  is integral. There are an index  $\alpha$ , a reductive group scheme  $\mathbf{G}_\alpha$  over  $R_\alpha$ , a principal  $\mathbf{G}_\alpha$ -bundle  $\mathcal{E}_\alpha$  over  $R_\alpha$ , and an element  $f_\alpha \in R_\alpha$  such that  $\mathbf{G} = \mathbf{G}_\alpha \times_{\text{Spec } R_\alpha} \text{Spec } R$ ,  $\mathcal{E}$  is isomorphic to  $\mathcal{E}_\alpha \times_{\text{Spec } R_\alpha} \text{Spec } R$  as principal  $\mathbf{G}$ -bundle,  $f$  is the image of  $f_\alpha$  under the homomorphism  $\varphi_\alpha : R_\alpha \rightarrow R$ ,  $\mathcal{E}_\alpha$  is trivial over  $(R_\alpha)_{f_\alpha}$ .

For each maximal ideal  $\mathfrak{m}_i$  in  $R$  ( $i = 1, \dots, n$ ) set  $\mathfrak{p}_i = \varphi_\alpha^{-1}(\mathfrak{m}_i)$ . The homomorphism  $\varphi_\alpha$  induces a homomorphism of semi-local rings  $(R_\alpha)_{\mathfrak{p}_1, \dots, \mathfrak{p}_n} \rightarrow R$ . By Lemma 3.3 the principal  $\mathbf{G}_\alpha$ -bundle  $\mathcal{E}_\alpha$  is trivial over  $(R_\alpha)_{\mathfrak{p}_1, \dots, \mathfrak{p}_n}$ . Whence the  $\mathbf{G}$ -bundle  $\mathcal{E}$  is trivial over  $R$ .  $\square$

#### 4. REDUCING THEOREM 2 TO THEOREM 3

Now we assume that Theorem 3 is true. Let  $k$ ,  $U$  and  $\mathbf{G}$  be as in Theorem 2. Let  $u_1, \dots, u_n$  be all the closed points of  $U$ . Let  $k(u_i)$  be the residue field of  $u_i$ . Consider the reduced closed subscheme  $\mathbf{u}$  of  $U$ , whose points are  $u_1, \dots, u_n$ . Thus

$$\mathbf{u} \cong \coprod_i \text{Spec } k(u_i).$$

Set  $\mathbf{G}_{\mathbf{u}} = \mathbf{G} \times_U \mathbf{u}$ . By  $\mathbf{G}_{u_i}$  we denote the fiber of  $\mathbf{G}$  over  $u_i$ ; it is a simple simply-connected algebraic group over  $k(u_i)$ .

**Proposition 4.1.** *Let  $Z \subset \mathbb{A}_U^1$  be a closed subscheme finite over  $U$ . There is a closed subscheme  $Y \subset \mathbb{A}_U^1$  which is étale and finite over  $U$  and such that*

- (i)  $\mathbf{G}_Y := \mathbf{G} \times_U Y$  is quasi-split,
- (ii)  $Y \cap Z = \emptyset$ ,
- (iii) for any closed point  $u \in U$  one has  $\text{Pic}(\mathbb{P}_u^1 - Y_u) = 0$ , where  $Y_u := \mathbb{P}_u^1 \cap Y$ .  
(Note that  $Y$  and  $Z$  are closed in  $\mathbb{P}_U^1$  since they are finite over  $U$ ).

*Proof.* For every  $u_i$  in  $\mathbf{u}$  choose a Borel subgroup  $\mathbf{B}_{u_i}$  in  $\mathbf{G}_{u_i}$ . The latter is possible since the fields  $k(u_i)$  are finite. Let  $\mathcal{B}$  be the  $U$ -scheme of Borel subgroup schemes of  $\mathbf{G}$ . It is a smooth projective  $U$ -scheme (see [DG, Cor. 3.5, Exp. XXVI]). The subgroup  $\mathbf{B}_{u_i}$  in  $\mathbf{G}_{u_i}$  is a  $k(u_i)$ -rational point  $b_i$  in the fibre of  $\mathcal{B}$  over the point  $u_i$ . Using a variant of Bertini theorem (see [Poo, Thm.1.2]), we can find a closed subscheme  $Y'$  of  $\mathcal{B}$  such that  $Y'$  is étale over  $U$  and all the  $b_i$ 's are in  $Y'$  (take an embedding of  $\mathcal{B}$  into a projective space  $\mathbb{P}_U^N$  and intersect  $\mathcal{B}$  with appropriately chosen family of hypersurfaces containing the points  $b_i$ . Arguing as in the proof of

[OP, Lemma 7.2], we get a scheme  $Y'$  finite and étale over  $U$ ). For any closed point  $u_i$  in  $U$  the fibre  $Y'_{u_i}$  of  $Y'$  over  $u_i$  contains a  $k(u_i)$ -rational point (it is the point  $b_i$ ).

To continue the proof of the Proposition we need the following

**Lemma 4.2.** *Let  $U$  be as in the Proposition. Let  $Z \subset \mathbb{A}_U^1$  be a closed subscheme finite over  $U$ . Let  $Y' \rightarrow U$  be a finite étale morphism such that for any closed point  $u_i$  in  $U$  the fibre  $Y'_{u_i}$  of  $Y'$  over  $u_i$  contains a  $k(u_i)$ -rational point. Then there are finite field extensions  $k_1$  and  $k_2$  of the finite field  $k$  such that*

- (i) *the degrees  $[k_1 : k]$  and  $[k_2 : k]$  are coprime,*
- (ii)  *$k(u_i) \otimes_k k_r$  is a field for  $r = 1$  and  $r = 2$ ,*
- (iii) *the degrees  $[k_1 : k]$  and  $[k_2 : k]$  are strictly greater than any of the degrees  $[k(z) : k]$ , where  $z$  runs over all closed points of  $Z$ ,*
- (iv) *there is a closed embedding of  $U$ -schemes  $Y'' = ((Y' \otimes_k k_1) \amalg (Y' \otimes_k k_2)) \xrightarrow{i} \mathbb{A}_U^1$ ,*
- (v) *for  $Y = i(Y'')$  one has  $Y \cap Z = \emptyset$ ,*
- (vi) *for any closed point  $u_i$  in  $U$  one has  $\text{Pic}(\mathbb{P}_{u_i}^1 - Y_{u_i}) = 0$ .*

To prove this Lemma note that it's easy to find field extensions  $k_1$  and  $k_2$  subjecting (i) to (iii). To satisfy (iv) it suffices to require that for any closed point  $u_i$  in  $U$  and for  $r = 1$  and  $r = 2$  the number of closed points in  $Y'_{u_i} \otimes_k k_r$  is the same as the number of closed points in  $Y'_{u_i}$ , and to require that for any integer  $n > 0$  and any closed point  $u_i$  in  $U$  the number of points  $y \in Y'_{u_i}$  with  $[k(y) : k(u_i)] = n$  is not more than the number of points  $x \in \mathbb{A}_{u_i}^1$  with  $[k(x) : k(u_i)] = n$ . Clearly, these requirements can be satisfied, which proves the item (iv).

The condition (v) holds for any closed  $U$ -embedding  $i : Y'' \hookrightarrow \mathbb{A}_U^1$  from item (iv), since the property (iii). The condition (vi) holds since the property (i).

Now complete the proof of Proposition 4.1. Take the  $U$ -scheme  $Y' \subset \mathbf{B}$  as in the beginning of the proof. This  $U$ -scheme  $Y'$  satisfies the assumption of Lemma 4.2. Take the closed subscheme  $Y$  of  $\mathbb{A}_U^1$  as in the item (v) of the Lemma. For this  $Y$  the conditions (ii) and (iii) of the Proposition are obviously satisfied. The condition (i) is satisfied too, since already it is satisfied for the  $U$ -scheme  $Y'$ . The Proposition follows.  $\square$

*Proof of Theorem 2.* Set  $Z := \{h = 0\} \cup s(U) \subset \mathbb{A}_U^1$ . Clearly,  $Z$  is finite over  $U$ . Since the principal  $\mathbf{G}$ -bundle  $\mathcal{E}_t$  is trivial over  $(\mathbb{A}_U^1)_h$  it is trivial over  $\mathbb{A}_U^1 - Z$ . Note that  $\{h = 0\}$  is closed in  $\mathbb{P}_U^1$  and finite over  $U$  because  $h$  is monic. Further,  $s(U)$  is also closed in  $\mathbb{P}_U^1$  and finite over  $U$  because it is a zero set of a degree one monic polynomial. Thus  $Z \subset \mathbb{P}_U^1$  is closed and finite over  $U$ .

Since the principal  $\mathbf{G}$ -bundle  $\mathcal{E}_t$  is trivial over  $(\mathbb{A}_U^1)_h$ , and  $\mathbf{G}$ -bundles can be glued in Zariski topology, there exists a principal  $\mathbf{G}$ -bundle  $\mathcal{G}$  over  $\mathbb{P}_U^1$  such that

- (i) its restriction to  $\mathbb{A}_U^1$  coincides with  $\mathcal{E}_t$ ;
- (ii) its restriction to  $\mathbb{P}_U^1 - Z$  is trivial.

Now choose  $Y$  in  $\mathbb{A}_U^1$  as in Proposition 4.1. Clearly,  $Y$  is finite étale over  $U$  and closed in  $\mathbb{P}_U^1$ . Moreover,  $Y \cap \{\infty\} \times U = \emptyset = Z \cap \{\infty\} \times U$  and  $Y \cap Z = \emptyset$ . Applying Theorem 3 with this choice of  $Y$  and  $Z$ , we see that the restriction of  $\mathcal{G}$  to  $\mathbb{P}_U^1 - Y$  is a trivial  $\mathbf{G}$ -bundle. Since  $s(U)$  is in  $\mathbb{A}_U^1 - Y$  and  $\mathcal{G}|_{\mathbb{A}_U^1}$  coincides with  $\mathcal{E}_t$ , we conclude that  $s^*\mathcal{E}_t$  is a trivial principal  $\mathbf{G}$ -bundle over  $U$ .  $\square$

## 5. PROOF OF THEOREM 3

We will be using notation from Theorem 3. Let  $\mathbf{u}$  be as in Section 4. For  $u \in \mathbf{u}$  set  $\mathbf{G}_u = \mathbf{G}|_u$ .

**Proposition 5.1.** *Let  $\mathcal{E}$  be a  $\mathbf{G}$ -bundle over  $\mathbb{P}_U^1$  such that  $\mathcal{E}|_{\mathbb{P}_u^1}$  is a trivial  $\mathbf{G}_u$ -bundle for all  $u \in \mathbf{u}$ . Assume that there exists a closed subscheme  $T$  of  $\mathbb{P}_U^1$  finite over  $U$  such that the restriction of  $\mathcal{E}$  to  $\mathbb{P}_U^1 - T$  is trivial and  $(\infty \times U) \cap T = \emptyset$ . Then  $\mathcal{E}$  is trivial.*

*Proof.* This follows from Theorem 9.6 of [PSV], since  $\mathcal{E}|_{(\infty \times U)}$  is a trivial  $\mathbf{G}$ -bundle.  $\square$

**5.1. An outline of the proof of Theorem 3.** Our proof of this Theorem almost literally coincides with the proof of [FP, Thm.3]. Our arguments are simpler at certain points.

An outline of the proof.

Denote by  $Y^h$  the henselization of the pair  $(\mathbb{A}_U^1, Y)$ , it is a scheme over  $\mathbb{A}_U^1$ . Let  $s : Y \rightarrow Y^h$  be the canonical closed embedding, see Section 5.2 for more details. Set  $\dot{Y}^h := Y^h - s(Y)$ . Let  $\mathcal{G}'$  be a  $\mathbf{G}$ -bundle over  $\mathbb{P}_U^1 - Y$ . Denote by  $\mathrm{Gl}(\mathcal{G}', \varphi)$  the  $\mathbf{G}$ -bundle over  $\mathbb{P}_U^1$  obtained by gluing  $\mathcal{G}'$  with the trivial  $\mathbf{G}$ -bundle  $\mathbf{G} \times_U Y^h$  via a  $\mathbf{G}$ -bundle isomorphism  $\varphi : \mathbf{G} \times_U \dot{Y}^h \rightarrow \mathcal{G}'|_{\dot{Y}^h}$ .

Note that the  $\mathbf{G}$ -bundle  $\mathcal{G}$  can be presented in the form  $\mathrm{Gl}(\mathcal{G}', \varphi)$ , where  $\mathcal{G}' = \mathcal{G}|_{\mathbb{P}_U^1 - Y}$ . The idea is to show that

*There is  $\alpha \in \mathbf{G}(\dot{Y}^h)$  such that the  $\mathbf{G}_u$ -bundle  $\mathrm{Gl}(\mathcal{G}', \varphi \circ \alpha)|_{\mathbb{P}_u^1}$  is trivial (here  $\alpha$  (\*) is regarded as an automorphism of the  $\mathbf{G}$ -bundle  $\mathbf{G} \times_U \dot{Y}^h$  given by the right translation by the element  $\alpha$ ).*

If we find  $\alpha$  satisfying condition (\*), then Proposition 5.1, applied to  $T = Y \cup Z$ , shows that the  $\mathbf{G}$ -bundle  $\mathrm{Gl}(\mathcal{G}', \varphi \circ \alpha)$  is trivial over  $\mathbb{P}_U^1$ . On the other hand, its restriction to  $\mathbb{P}_U^1 - Y$  coincides with the  $\mathbf{G}$ -bundle  $\mathcal{G}' = \mathcal{G}|_{\mathbb{P}_U^1 - Y}$ . Thus  $\mathcal{G}|_{\mathbb{P}_U^1 - Y}$  is a trivial  $\mathbf{G}$ -bundle.

To prove (\*) it suffices to show that

- (i) the bundle  $\mathcal{G}|_{\mathbb{P}_u^1 - Y_u}$  is trivial;
- (ii) each element  $\gamma_u \in \mathbf{G}_u(\dot{Y}_u^h)$  can be written in the form

$$\alpha|_{\dot{Y}_u^h} \cdot \beta_u|_{\dot{Y}_u^h}$$

for certain elements  $\alpha \in \mathbf{G}(\dot{Y}^h)$  and  $\beta_u \in \mathbf{G}_u(Y_u^h)$ .

A realization of this plan in details is given below in the paper.

**5.2. Henselization of affine pairs.** We will use the theory of henselian pairs and, in particular, a notion of a henselization  $A_I^h$  of a commutative ring  $A$  at an ideal  $I$  (see Appendix and [Gab, Section 0]). We refer to [FP, subsection 5.2] for the geometric counterpart. Let  $S = \mathrm{Spec} A$  be a scheme and  $T = \mathrm{Spec}(A/I)$  be a closed subscheme. Let  $(T^h, \pi : T^h \rightarrow S, s : T \rightarrow T^h)$  be the henselization of the pair  $(S, T)$  (cf. Definition A.3). By definition the scheme  $T^h$  is affine and the composite morphism  $\pi \circ s : T \rightarrow S$  is the closed embedding  $T \hookrightarrow S$ . Recall that the pair  $(T^h, s(T))$  is henselian, which means that for any affine étale morphism  $\pi : Z \rightarrow T^h$ , any section  $\sigma$  of  $\pi$  over  $s(T)$  uniquely extends to a section of  $\pi$  over  $T^h$ . It is known that  $\pi^{-1}(T) = s(T)$ .

In the notation of [Gab, Section 0] we have  $T^h = \text{Spec } A_I^h$ ,  $\pi : T^h \rightarrow S$  is induced by the structure of  $A$ -algebra on  $A_I^h$ .

*Recall three properties of henselization of affine pairs*

(i) Let  $T$  be a semi-local scheme. Then the henselization commutes with restriction to closed subschemes. In more details, if  $S' \subset S$  is a closed subscheme, then there is a natural morphism  $(T \times_S S')^h \rightarrow T^h \times_S S'$ . This morphism is an isomorphism and the canonical section  $s' : T \times_S S' \rightarrow (T \times_S S')^h$  coincides under this identification with

$$s \times_S \text{Id}_{S'} : T \times_S S' \rightarrow T^h \times_S S'.$$

(ii) If  $T = \coprod_i T_i$  is a disjoint union, then  $T^h = \coprod_i T_i^h$ .

(iii) If we replace in a pair  $(S, T)$  the scheme  $S$  by an étale affine neighborhood of  $T$ , then the  $(T^h, \pi, s)$  remains the same. In more details, given a pair  $(S, T)$  as above we write temporarily  $(S_T^{\text{hen}}, \pi_{S,T}, s_{S,T})$  for  $(T^h, \pi, s)$ . If  $p : W \rightarrow S$  is an étale morphism and  $t : T \hookrightarrow W$  is such that  $p \circ t : T \hookrightarrow S$  coincides with the closed embedding  $T$  into  $S$ , then there is a canonical isomorphism  $\rho : W_T^{\text{hen}} \rightarrow S_T^{\text{hen}}$  of the  $S$ -schemes  $(W_T^{\text{hen}}, \pi_{W,T})$  and  $(S_T^{\text{hen}}, \pi_{S,T})$  such that  $\rho \circ s_{W,T} = s_{S,T}$ .

**5.3. Gluing principal  $\mathbf{G}$ -bundles.** Recall that  $U = \text{Spec } R$ , where  $R$  is the semi-local ring of finitely many closed points on an irreducible  $k$ -smooth affine variety over a finite field  $k$ . Also,  $\mathbf{G}$  is a simple simply-connected group scheme over  $U$ , and  $Y$  is a closed subscheme of  $\mathbb{P}_U^1$  finite and étale over  $U$ .

We will assume below in the preprint that  $Y \subset \mathbb{A}_U^1$  (as in the hypotheses of Theorem 3). Let  $(Y^h, \pi, s)$  be the henselization of the pair  $(\mathbb{A}_U^1, Y)$  and let  $\dot{Y}^h = Y^h - s(Y)$  and let  $\text{in} : \mathbb{A}_U^1 \hookrightarrow \mathbb{P}_U^1$  be the open inclusion.

**Proposition 5.2.** [FP] *The schemes  $Y^h$  and  $\dot{Y}^h$  are affine.*

Let us make a general remark. Let  $\mathcal{F}$  be a  $\mathbf{G}$ -bundle over a  $U$ -scheme  $T$ . By definition, a trivialization of  $\mathcal{F}$  is a  $\mathbf{G}$ -equivariant isomorphism  $\mathbf{G} \times_U T \rightarrow \mathcal{F}$ . Equivalently, it is a section of the projection  $\mathcal{F} \rightarrow T$ . If  $\varphi$  is such a trivialization and  $f : T' \rightarrow T$  is a  $U$ -morphism, we get a trivialization  $f^* \varphi$  of  $f^* \mathcal{F}$ . Sometimes we denote this trivialization by  $\varphi|_{T'}$ . We also sometimes call a trivialization of  $f^* \mathcal{F}$  a *trivialization of  $\mathcal{F}$  on  $T'$* .

The main cartesian square we will work with is

$$(1) \quad \begin{array}{ccc} \dot{Y}^h & \longrightarrow & Y^h \\ \downarrow & & \downarrow \text{in} \circ \pi \\ \mathbb{P}_U^1 - Y & \longrightarrow & \mathbb{P}_U^1. \end{array}$$

Let  $\mathcal{A}$  be the category of pairs  $(\mathcal{E}, \psi)$ , where  $\mathcal{E}$  is a  $\mathbf{G}$ -bundle on  $\mathbb{P}_U^1$ ,  $\psi$  is a trivialization of  $\mathcal{E}|_{Y^h} := (\text{in} \circ \pi)^* \mathcal{E}$ . A morphism between  $(\mathcal{E}, \psi)$  and  $(\mathcal{E}', \psi')$  is an isomorphism  $\mathcal{E} \rightarrow \mathcal{E}'$  compatible with trivializations.

Similarly, let  $\mathcal{B}$  be the category of pairs  $(\mathcal{E}, \psi)$ , where  $\mathcal{E}$  is a  $\mathbf{G}$ -bundle on  $\mathbb{P}_U^1 - Y$ ,  $\psi$  is a trivialization of  $\mathcal{E}|_{\dot{Y}^h}$ .

Consider the restriction functor  $\Psi : \mathcal{A} \rightarrow \mathcal{B}$ .

**Proposition 5.3.** [FP] *The functor  $\Psi$  is an equivalence of categories.*



*Construction 5.4.* [FP] By Proposition 5.3 we can choose a functor quasi-inverse to  $\Psi$ . Fix such a functor  $\Theta$ . Let  $\Lambda$  be the forgetful functor from  $\mathcal{A}$  to the category of  $\mathbf{G}$ -bundles over  $\mathbb{P}_U^1$ . For  $(\mathcal{E}, \psi) \in \mathcal{B}$  set

$$\mathrm{Gl}(\mathcal{E}, \psi) = \Lambda(\Theta(\mathcal{E}, \psi)).$$

Note that  $\mathrm{Gl}(\mathcal{E}, \psi)$  comes with a canonical trivialization over  $Y^h$ .

Conversely, if  $\mathcal{E}$  is a principal  $\mathbf{G}$ -bundle over  $\mathbb{P}_U^1$  such that its restriction to  $Y^h$  is trivial, then  $\mathcal{E}$  can be represented as  $\mathrm{Gl}(\mathcal{E}', \psi)$ , where  $\mathcal{E}' = \mathcal{E}|_{\mathbb{P}_U^1 - Y}$ ,  $\psi$  is a trivialization of  $\mathcal{E}'$  on  $\dot{Y}^h$ .

Let  $\mathbf{u}$  be as in Section 4,  $Y_{\mathbf{u}} := Y \times_U \mathbf{u}$ . Let  $(Y_{\mathbf{u}}^h, \pi_{\mathbf{u}}, s_{\mathbf{u}})$  be the henselization of  $(\mathbb{A}_{\mathbf{u}}^1, Y_{\mathbf{u}})$ . Using property (i) of henselization, we get  $Y_{\mathbf{u}}^h = Y^h \times_U \mathbf{u}$ . Thus we have a natural closed embedding  $Y_{\mathbf{u}}^h \rightarrow Y^h$ . Set  $\dot{Y}_{\mathbf{u}}^h = Y_{\mathbf{u}}^h - s_{\mathbf{u}}(Y_{\mathbf{u}})$ . We get a closed embedding

$$(2) \quad \dot{Y}_{\mathbf{u}}^h \hookrightarrow \dot{Y}^h.$$

Thus the pull-back of the cartesian square (1) by means of the closed embedding  $\mathbf{u} \hookrightarrow U$  has the form

$$\begin{array}{ccc} \dot{Y}_{\mathbf{u}}^h & \longrightarrow & Y_{\mathbf{u}}^h \\ \downarrow & & \downarrow \mathrm{in}_{\mathbf{u}} \circ \pi_{\mathbf{u}} \\ \mathbb{P}_{\mathbf{u}}^1 - Y_{\mathbf{u}} & \longrightarrow & \mathbb{P}_{\mathbf{u}}^1, \end{array}$$

where  $\mathrm{in}_{\mathbf{u}} : \mathbb{A}_{\mathbf{u}}^1 \rightarrow \mathbb{P}_{\mathbf{u}}^1$ .

Similarly to the above, we can define categories  $\mathcal{A}_{\mathbf{u}}$  and  $\mathcal{B}_{\mathbf{u}}$  and an equivalence of categories  $\Psi_{\mathbf{u}} : \mathcal{A}_{\mathbf{u}} \rightarrow \mathcal{B}_{\mathbf{u}}$ . Let  $\Theta_{\mathbf{u}}$  be a functor quasi-inverse to  $\Psi_{\mathbf{u}}$  and  $\Lambda_{\mathbf{u}}$  be the forgetful functor from  $\mathcal{A}_{\mathbf{u}}$  to the category of  $\mathbf{G}_{\mathbf{u}}$ -bundles over  $\mathbb{P}_{\mathbf{u}}^1$ . Let  $\mathcal{E}_{\mathbf{u}}$  be a principal  $\mathbf{G}_{\mathbf{u}}$ -bundle over  $\mathbb{P}_{\mathbf{u}}^1 - Y_{\mathbf{u}}$  and  $\psi_{\mathbf{u}}$  be a trivialization of  $\mathbf{G}_{\mathbf{u}}$  on  $\dot{Y}_{\mathbf{u}}^h$ . Set

$$\mathrm{Gl}_{\mathbf{u}}(\mathcal{E}_{\mathbf{u}}, \psi_{\mathbf{u}}) = \Lambda_{\mathbf{u}}(\Theta_{\mathbf{u}}(\mathcal{E}_{\mathbf{u}}, \psi_{\mathbf{u}})).$$

**Lemma 5.5.** [FP] *Let  $(\mathcal{E}, \psi) \in \mathcal{B}$ , and let  $\mathrm{Gl}(\mathcal{E}, \psi)$  be the  $\mathbf{G}$ -bundle obtained by Construction 5.4. Then*

$$\mathrm{Gl}_{\mathbf{u}}(\mathcal{E}|_{\mathbb{P}_{\mathbf{u}}^1 - Y_{\mathbf{u}}}, \psi|_{\dot{Y}_{\mathbf{u}}^h}) \text{ and } \mathrm{Gl}(\mathcal{E}, \psi)|_{\mathbb{P}_{\mathbf{u}}^1}$$

*are isomorphic as  $\mathbf{G}_{\mathbf{u}}$ -bundles over  $\mathbb{P}_{\mathbf{u}}^1$ .*

**Lemma 5.6.** [FP] *For any  $(\mathcal{E}_{\mathbf{u}}, \psi_{\mathbf{u}}) \in \mathcal{B}_{\mathbf{u}}$  and any  $\beta_{\mathbf{u}} \in \mathbf{G}_{\mathbf{u}}(Y_{\mathbf{u}}^h)$  the  $\mathbf{G}_{\mathbf{u}}$ -bundles*

$$\mathrm{Gl}_{\mathbf{u}}(\mathcal{E}_{\mathbf{u}}, \psi_{\mathbf{u}}) \text{ and } \mathrm{Gl}_{\mathbf{u}}(\mathcal{E}_{\mathbf{u}}, \psi_{\mathbf{u}} \circ \beta_{\mathbf{u}}|_{\dot{Y}_{\mathbf{u}}^h})$$

*are isomorphic (here  $\beta_{\mathbf{u}}|_{\dot{Y}_{\mathbf{u}}^h}$  is regarded as an automorphism of the  $\mathbf{G}_{\mathbf{u}}$ -bundle  $\mathbf{G}_{\mathbf{u}} \times_{\mathbf{u}} \dot{Y}_{\mathbf{u}}^h$  given by the right translation by  $\beta_{\mathbf{u}}|_{\dot{Y}_{\mathbf{u}}^h}$ ).*

**5.4. Proof of Theorem 3: presentation of  $\mathcal{G}$  in the form  $\mathrm{Gl}(\mathcal{G}', \varphi)$ .** Let  $U$ ,  $\mathbf{G}$ ,  $Z$ ,  $Y$  and  $\mathcal{G}$  be as in Theorem 3.

**Proposition 5.7.** [FP] *The  $\mathbf{G}$ -bundle  $\mathcal{G}$  over  $\mathbb{P}_U^1$  is of the form  $\mathrm{Gl}(\mathcal{G}', \varphi)$  for the  $\mathbf{G}$ -bundle  $\mathcal{G}' := \mathcal{G}|_{\mathbb{P}_U^1 - Y}$  and a trivialization  $\varphi$  of  $\mathcal{G}'$  over  $\dot{Y}^h$ .*

*Proof.* In view of Construction 5.4, it is enough to prove that the restriction of the principal  $\mathbf{G}$ -bundle  $\mathcal{G}$  to  $Y^h$  is trivial. Let us choose a closed subscheme  $Z' \subset \mathbb{A}_U^1$  such that  $Z'$  contains  $Z$ ,  $Z' \cap Y = \emptyset$ , and  $\mathbb{A}_U^1 - Z'$  is affine. Then  $\mathbb{A}_U^1 - Z'$  is an affine neighborhood of  $Y$ . By the property (iii) from subsection 5.2 the henselization of

the pair  $(\mathbb{A}_U^1 - Z', Y)$  coincides with the henselization of the pair  $(\mathbb{A}_U^1, Y)$ . Since  $\mathcal{G}$  is trivial over  $\mathbb{A}_U^1 - Z'$ , its pull-back to  $Y^h$  is trivial too. The proposition is proved.  $\square$

*Our aim is to modify the trivialization  $\varphi$  via an element*

$$\alpha \in \mathbf{G}(\dot{Y}^h)$$

*so that the  $\mathbf{G}$ -bundle  $\mathrm{Gl}(\mathcal{G}', \varphi \circ \alpha)$  becomes trivial over  $\mathbb{P}_U^1$ .*

**5.5. Proof of Theorem 3: proof of property (i) from the outline.** Now we are able to prove property (i) from the outline of the proof. In fact, we will prove the following modification of [FP, Lemma 5.11].

**Lemma 5.8.** *Let  $\mathrm{Gl}(\mathcal{G}', \varphi)$  be the presentation of the  $\mathbf{G}$ -bundle  $\mathcal{G}$  over  $\mathbb{P}_U^1$  given in Proposition 5.7. Set  $\varphi_{\mathbf{u}} := \varphi|_{\dot{Y}_{\mathbf{u}}^h}$ . Then there is  $\gamma_{\mathbf{u}} \in \mathbf{G}_{\mathbf{u}}(\dot{Y}_{\mathbf{u}}^h)$  such that the  $\mathbf{G}_{\mathbf{u}}$ -bundle  $\mathrm{Gl}_{\mathbf{u}}(\mathcal{G}'|_{\mathbb{P}_{\mathbf{u}}^1 - Y_{\mathbf{u}}}, \varphi_{\mathbf{u}} \circ \gamma_{\mathbf{u}})$  is trivial.*

*Proof.* We show first that  $\mathcal{G}|_{\mathbb{P}_{\mathbf{u}}^1 - Y_{\mathbf{u}}}$  is trivial. One has

$$\mathbb{P}_{\mathbf{u}}^1 = \coprod_{u \in \mathbf{u}} \mathbb{P}_u^1$$

For  $u \in \mathbf{u}$  set  $Y_u := Y \times_U u$ ,  $\mathbf{G}_u := \mathbf{G} \times_U u$ , and  $\mathcal{G}_u := \mathcal{G} \times_U u$ .

Take  $u \in \mathbf{u}$ . By our assumption on  $Y$ ,  $\mathrm{Pic}(\mathbb{P}_u^1 - Y_u) = 0$ . The  $\mathbf{G}_u$ -bundle  $\mathcal{G}_u$  is trivial over  $\mathbb{A}_u^1 - Z_u$ . Thus, by [Gill, Corollary 3.10(a)], it is trivial over  $\mathbb{P}_u^1 - Y_u$ .

We see that  $\mathcal{G}'|_{\mathbb{P}_{\mathbf{u}}^1 - Y_{\mathbf{u}}} = \mathcal{G}|_{\mathbb{P}_{\mathbf{u}}^1 - Y_{\mathbf{u}}}$  is trivial. Choosing a trivialization, we may identify  $\varphi_{\mathbf{u}}$  with an element of  $\mathbf{G}_{\mathbf{u}}(\dot{Y}_{\mathbf{u}}^h)$ . Set  $\gamma_{\mathbf{u}} = \varphi_{\mathbf{u}}^{-1}$ . By the very choice of  $\gamma_{\mathbf{u}}$  the  $\mathbf{G}_{\mathbf{u}}$ -bundle  $\mathrm{Gl}_{\mathbf{u}}(\mathcal{G}'|_{\mathbb{P}_{\mathbf{u}}^1 - Y_{\mathbf{u}}}, \varphi_{\mathbf{u}} \circ \gamma_{\mathbf{u}})$  is trivial.  $\square$

**5.6. Proof of Theorem 3: reduction to property (ii) from the outline.** The aim of this section is to deduce Theorem 3 from the following

**Proposition 5.9.** [FP] *Each element  $\gamma_{\mathbf{u}} \in \mathbf{G}_{\mathbf{u}}(\dot{Y}_{\mathbf{u}}^h)$  can be written in the form*

$$\alpha|_{\dot{Y}_{\mathbf{u}}^h} \cdot \beta_{\mathbf{u}}|_{\dot{Y}_{\mathbf{u}}^h}$$

*for certain elements  $\alpha \in \mathbf{G}(\dot{Y}^h)$  and  $\beta_{\mathbf{u}} \in \mathbf{G}_{\mathbf{u}}(Y_{\mathbf{u}}^h)$ .*

*Deduction of Theorem 3 from Proposition 5.9.* [FP] Let  $\mathrm{Gl}(\mathcal{G}', \varphi)$  be the presentation of the  $\mathbf{G}$ -bundle  $\mathcal{G}$  from Proposition 5.7. Let  $\gamma_{\mathbf{u}} \in \mathbf{G}_{\mathbf{u}}(\dot{Y}_{\mathbf{u}}^h)$  be the element from Lemma 5.8. Let  $\alpha \in \mathbf{G}(\dot{Y}^h)$  and  $\beta_{\mathbf{u}} \in \mathbf{G}_{\mathbf{u}}(Y_{\mathbf{u}}^h)$  be the elements from Proposition 5.9. Set

$$\mathcal{G}^{new} = \mathrm{Gl}(\mathcal{G}', \varphi \circ \alpha).$$

*Claim.* The  $\mathbf{G}$ -bundle  $\mathcal{G}^{new}$  is trivial over  $\mathbb{P}_U^1$ .

Indeed, by Lemmas 5.5 and 5.6 one has a chain of isomorphisms of  $\mathbf{G}_{\mathbf{u}}$ -bundles

$$\begin{aligned} \mathcal{G}^{new}|_{\mathbb{P}_{\mathbf{u}}^1} &\cong \mathrm{Gl}_{\mathbf{u}}(\mathcal{G}'|_{\mathbb{P}_{\mathbf{u}}^1 - Y_{\mathbf{u}}}, \varphi_{\mathbf{u}} \circ \alpha|_{\dot{Y}_{\mathbf{u}}^h}) \cong \\ &\mathrm{Gl}_{\mathbf{u}}(\mathcal{G}'|_{\mathbb{P}_{\mathbf{u}}^1 - Y_{\mathbf{u}}}, \varphi_{\mathbf{u}} \circ \alpha|_{\dot{Y}_{\mathbf{u}}^h} \circ \beta_{\mathbf{u}}|_{\dot{Y}_{\mathbf{u}}^h}) = \mathrm{Gl}_{\mathbf{u}}(\mathcal{G}'|_{\mathbb{P}_{\mathbf{u}}^1 - Y_{\mathbf{u}}}, \varphi_{\mathbf{u}} \circ \gamma_{\mathbf{u}}), \end{aligned}$$

which is trivial by the choice of  $\gamma_{\mathbf{u}}$ . The  $\mathbf{G}$ -bundles  $\mathcal{G}|_{\mathbb{P}_U^1 - Y}$  and  $\mathcal{G}^{new}|_{\mathbb{P}_U^1 - Y}$  coincide by the very construction of  $\mathcal{G}^{new}$ . By Proposition 5.1, applied to  $T = Z \cup Y$ , the  $\mathbf{G}$ -bundle  $\mathcal{G}^{new}$  is trivial. Whence the claim.

The claim above implies that the  $\mathbf{G}$ -bundle  $\mathcal{G}|_{\mathbb{P}_U^1 - Y} = \mathcal{G}^{new}|_{\mathbb{P}_U^1 - Y}$  is trivial. Theorem 3 is proved.  $\square$

**5.7. End of proof of Theorem 3: proof of property (ii) from the outline.**  
*In the remaining part of Section 5 we will prove Proposition 5.9. This will complete the proof of Theorem 3.*

By our assumption on  $Y$ , the group scheme  $\mathbf{G}_Y = \mathbf{G} \times_U Y$  is **quasi-split**. Thus we can and will choose a Borel subgroup scheme  $\mathbf{B}^+$  in  $\mathbf{G}_Y$ .

Since  $Y$  is an affine scheme, by [DG, Exp. XXVI, Cor. 2.3, Th 4.3.2(a)] there is an opposite to  $\mathbf{B}^+$  Borel subgroup scheme  $\mathbf{B}^-$  in  $\mathbf{G}_Y$ . Let  $\mathbf{U}^+$  be the unipotent radical of  $\mathbf{B}^+$ , and let  $\mathbf{U}^-$  be the unipotent radical of  $\mathbf{B}^-$ .

**Definition 5.10.** We will write  $\mathbf{E}$  for the functor, sending a  $Y$ -scheme  $T$  to the subgroup  $\mathbf{E}(T)$  of the group  $\mathbf{G}_Y(T) = \mathbf{G}(T)$  generated by the subgroups  $\mathbf{U}^+(T)$  and  $\mathbf{U}^-(T)$  of the group  $\mathbf{G}_Y(T) = \mathbf{G}(T)$ .

**Lemma 5.11.** *The functor  $\mathbf{E}$  has the property that for every closed subscheme  $S$  in an affine  $Y$ -scheme  $T$  the induced map  $\mathbf{E}(T) \rightarrow \mathbf{E}(S)$  is surjective.*

*Proof.* The restriction maps  $\mathbf{U}^\pm(T) \rightarrow \mathbf{U}^\pm(S)$  are surjective, since  $\mathbf{U}^\pm$  are isomorphic to vector bundles as  $Y$ -schemes (see [DG, Exp. XXVI, Cor. 2.5]).  $\square$

Recall that  $(Y^h, \pi, s)$  is the henselization of the pair  $(\mathbb{A}_U^1, Y)$ . Also,  $in : \mathbb{A}_U^1 \rightarrow \mathbb{P}_U^1$  is the embedding. Denote the projection  $\mathbb{A}_U^1 \rightarrow U$  by  $pr$  and the projection  $\mathbb{A}_Y^1 \rightarrow Y$  by  $pr_Y$ .

**Lemma 5.12.** [FP] *There is a morphism  $r : Y^h \rightarrow Y$  making the following diagram commutative*

$$(3) \quad \begin{array}{ccc} Y^h & \xrightarrow{r} & Y \\ in \circ \pi \downarrow & & \downarrow pr|_Y \\ \mathbb{P}_U^1 & \xrightarrow{pr} & U \end{array}$$

and such that  $r \circ s = \text{Id}_Y$ .

We view  $Y^h$  as a  $Y$ -scheme via  $r$ . Thus various subschemes of  $Y^h$  also become  $Y$ -schemes. In particular,  $\dot{Y}^h$  and  $\dot{Y}_{\mathbf{u}}^h$  are  $Y$ -schemes, and we can consider

$$\mathbf{E}(\dot{Y}^h) \subset \mathbf{G}(\dot{Y}^h) \quad \text{and} \quad \mathbf{E}(\dot{Y}_{\mathbf{u}}^h) \subset \mathbf{G}(\dot{Y}_{\mathbf{u}}^h) = \mathbf{G}_{\mathbf{u}}(\dot{Y}_{\mathbf{u}}^h).$$

**Lemma 5.13.**

$$\mathbf{G}_{\mathbf{u}}(\dot{Y}_{\mathbf{u}}^h) = \mathbf{E}(\dot{Y}_{\mathbf{u}}^h) \mathbf{G}_{\mathbf{u}}(Y_{\mathbf{u}}^h).$$

*Proof.* Firstly, one has  $Y_{\mathbf{u}} = \coprod_{u \in \mathbf{u}} \coprod_{y \in Y_u} y$ . (Note that  $Y_u$  is a finite scheme.) Thus by property (ii) of henselization, we have

$$Y_{\mathbf{u}}^h = \coprod_{u \in \mathbf{u}} \coprod_{y \in Y_u} y^h, \quad \dot{Y}_{\mathbf{u}}^h = \coprod_{u \in \mathbf{u}} \coprod_{y \in Y_u} \dot{y}^h,$$

where  $(y^h, \pi_y, s_y)$  is the henselization of the pair  $(\mathbb{A}_u^1, y)$ ,  $\dot{y}^h := y^h - s_y(y)$ . We see that  $y^h$  and  $\dot{y}^h$  are subschemes of  $Y^h$ , so we can view them as  $Y$ -schemes, and  $\mathbf{G}_{y^h} := \mathbf{G}_Y \times_Y y^h$  is **quasi-split**. Also,  $\mathbf{E}(\dot{y}^h)$  makes sense as a subgroup of  $\mathbf{G}(\dot{y}^h) = \mathbf{G}_u(\dot{y}^h) = \mathbf{G}_{y^h}(\dot{y}^h)$ .

One has

$$\begin{aligned}\mathbf{G}_{\mathbf{u}}(\dot{Y}_{\mathbf{u}}^h) &= \prod_{u \in \mathbf{u}} \prod_{y \in Y_u} \mathbf{G}_u(\dot{y}^h) = \prod_{u \in \mathbf{u}} \prod_{y \in Y_u} \mathbf{G}_{y^h}(\dot{y}^h), \\ \mathbf{E}(\dot{Y}_{\mathbf{u}}^h) &= \prod_{u \in \mathbf{u}} \prod_{y \in Y_u} \mathbf{E}(\dot{y}^h), \\ \mathbf{G}_{\mathbf{u}}(Y_{\mathbf{u}}^h) &= \prod_{u \in \mathbf{u}} \prod_{y \in Y_u} \mathbf{G}_u(y^h) = \prod_{u \in \mathbf{u}} \prod_{y \in Y_u} \mathbf{G}_{y^h}(y^h).\end{aligned}$$

Thus it suffices for each  $u \in \mathbf{u}$  and each  $y \in Y_u$  to check the equality

$$\mathbf{G}_{y^h}(\dot{y}^h) = \mathbf{E}(\dot{y}^h) \mathbf{G}_{y^h}(y^h).$$

This equality holds by Fait 4.3 and Lemma 4.5 of [Gil2]. In fact,  $y^h = \text{Spec } \mathcal{O}$ , where  $\mathcal{O} = k(u)[t]_{\mathfrak{m}_y}^h$  is a henselian discrete valuation ring, and  $\mathfrak{m}_y \subset k(u)[t]$  is the maximal ideal defining the point  $y \in \mathbb{A}_u^1$ . Further,  $\dot{y}^h = \text{Spec } L$ , where  $L$  is the fraction field of  $\mathcal{O}$ . The lemma is proved.  $\square$

We have the closed embedding (2) and the scheme  $\dot{Y}^h$  is affine by Proposition 5.2. Recall that we regard  $\dot{Y}^h$  as a  $Y$ -scheme via the morphism  $r|_{\dot{Y}^h}$ . Thus by Lemma 5.11 the restriction map  $\mathbf{E}(\dot{Y}^h) \rightarrow \mathbf{E}(\dot{Y}_{\mathbf{u}}^h)$  is surjective. Since  $\mathbf{E}(\dot{Y}^h) \subset \mathbf{G}(\dot{Y}^h)$ , the proposition 5.9 follows. *This completes the proof of Theorem 3.*

## 6. AN APPLICATION

The following result is a straightforward consequence of Theorem 1 and an exact sequence for étale cohomology. Recall that by our definition a reductive group scheme has geometrically connected fibres.

**Theorem 4.** *Let  $R$  be as in Theorem 1 and  $\mathbf{G}$  be a reductive  $R$ -group scheme. Let  $\mu : \mathbf{G} \rightarrow \mathbf{T}$  be a group scheme morphism to an  $R$ -torus  $\mathbf{T}$  such that  $\mu$  is locally in the étale topology on  $\text{Spec } R$  surjective. Assume further that the  $R$ -group scheme  $\mathbf{H} := \text{Ker}(\mu)$  is reductive. Let  $K$  be the fraction field of  $R$ . Then the group homomorphism*

$$\mathbf{T}(R)/\mu(\mathbf{G}(R)) \rightarrow \mathbf{T}(K)/\mu(\mathbf{G}(K))$$

*is injective.*

This theorem extends all the known results of this form proven in [CTO], [PS1], [Zai], [OPZ].

## APPENDIX A. [FP]

For a commutative ring  $A$  we denote by  $\text{Rad}(A)$  its Jacobson ideal. The following definition one can find in [Gab, Section 0].

**Definition A.1.** If  $I$  is an ideal in a commutative ring  $A$ , then the pair  $(A, I)$  is called *henselian*, if  $I \subset \text{Rad}(A)$  and for every two relatively prime monic polynomials  $\bar{g}, \bar{h} \in \bar{A}[t]$ , where  $\bar{A} = A/I$ , and monic lifting  $f \in A[t]$  of  $\bar{g}\bar{h}$ , there exist monic liftings  $g, h \in A[t]$  such that  $f = gh$ . (Two polynomials are called relatively prime, if they generate the unit ideal.)

**Lemma A.2.** [FP] *Let  $(A, I)$  be a henselian pair with a semi-local ring  $A$  and  $J \subset A$  be an ideal. Then the pair  $(A/J, (I + J)/J)$  is henselian.*

The following definition one can find in [Gab, Section 0].

**Definition A.3.** The henselization of any pair  $(A, I)$  is the pair  $(A_I^h, I^h)$  (over  $(A, I)$ ) defined as follows

$$(A_I^h, I^h) := \text{the filtered inductive limit over the category } \mathcal{N} \text{ of } (A', \text{Ker}(\sigma)),$$

where  $\mathcal{N}$  is the filtered category of pairs  $(A', \sigma)$  such that  $A'$  is an étale  $A$ -algebra and  $\sigma \in \text{Hom}_{A\text{-alg}}(A', A/I)$ .

## REFERENCES

- [Che] Vladimir Chernousov. Variations on a theme of groups splitting by a quadratic extension and Grothendieck–Serre conjecture for group schemes  $F_4$  with trivial  $g_3$  invariant. *Doc. Math.*, (Extra volume: Andrei A. Suslin sixtieth birthday):147–169, 2010.
- [CTO] Jean-Louis Colliot-Thélène and Manuel Ojanguren. Espaces principaux homogènes localement triviaux. *Inst. Hautes Études Sci. Publ. Math.*, (75):97–122, 1992.
- [CTS] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. Principal homogeneous spaces under flasque tori: applications. *J. Algebra*, 106(1):148–205, 1987.
- [DG] Michel Demazure and Alexander Grothendieck. *Schémas en groupes. III: Structure des schémas en groupes réductifs*. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153. Springer-Verlag, Berlin, 1970.
- [FP] Fedorov, R.; Panin, I. A proof of Grothendieck–Serre conjecture on principal bundles over a semilocal regular ring containing an infinite field, Preprint, April 2013, <http://www.arxiv.org/abs/1211.2678v2>.
- [Gab] Ofer Gabber. Affine analog of the proper base change theorem. *Israel J. Math.*, 87(1-3):325–335, 1994.
- [Gil1] Philippe Gille. Torseurs sur la droite affine. *Transform. Groups*, 7(3):231–245, 2002.
- [Gil2] Philippe Gille. Le problème de Kneser-Tits. *Astérisque*, (326):Exp. No. 983, vii, 39–81 (2010), 2009. Séminaire Bourbaki. Vol. 2007/2008.
- [Gro1] Alexander Grothendieck. Torsion homologique et sections rationnelles. In *Anneaux de Chow et applications, Séminaire Claude Chevalley*, number 3. Paris, 1958.
- [Gro2] Alexander Grothendieck. Le groupe de Brauer. II. Théorie cohomologique. In *Dix Exposés sur la Cohomologie des Schémas*, pages 67–87. North-Holland, Amsterdam, 1968.
- [Gro3] Alexander Grothendieck. Technique de descente et théorèmes d’existence en géométrie algébrique. I. Généralités. Descente par morphismes fidèlement plats. In *Séminaire Bourbaki, Vol. 5, Exp. No. 190.*, pages 299–327. Soc. Math. France, Paris, 1995.
- [Nis1] Yevsey Nisnevich. Espaces homogènes principaux rationnellement triviaux et arithmétique des schémas en groupes réductifs sur les anneaux de Dedekind. *C. R. Acad. Sci. Paris Sér. I Math.*, 299(1):5–8, 1984.
- [Nis2] Yevsey Nisnevich. Rationally trivial principal homogeneous spaces, purity and arithmetic of reductive group schemes over extensions of two-dimensional regular local rings. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(10):651–655, 1989.
- [Oja1] Manuel Ojanguren. Quadratic forms over regular rings. *J. Indian Math. Soc. (N.S.)*, 44(1-4):109–116 (1982), 1980.
- [Oja2] Manuel Ojanguren. Unités représentées par des formes quadratiques ou par des normes réduites. In *Algebraic K-theory, Part II (Oberwolfach, 1980)*, volume 967 of *Lecture Notes in Math.*, pages 291–299. Springer, Berlin, 1982.
- [OP] Manuel Ojanguren and Ivan Panin. Rationally trivial Hermitian spaces are locally trivial. *Math. Z.*, 237(1):181–198, 2001.
- [OPZ] M. Ojanguren, I. Panin, and K. Zainoulline. On the norm principle for quadratic forms. *J. Ramanujan Math. Soc.*, 19(4):289–300, 2004.
- [Pa1] I. Panin. On Grothendieck–Serre’s conjecture concerning principal  $G$ -bundles over reductive group schemes containing a finite field: I. *ArXiv e-prints, 0905.1418v3*, April 2013.
- [Pa2] Ivan Panin. On Grothendieck–Serre’s conjecture concerning principal  $G$ -bundles over reductive group containing a finite field schemes:II. *ArXiv e-prints, 0905.1423v3*, April 2013.

- [Pan1] *Panin, I.* On Grothendieck-Serre conjecture concerning principal  $G$ -bundles over regular semi-local domains containing a finite field: I, Preprint, May 2014.
- [Pan2] *Panin, I.* On Grothendieck-Serre conjecture concerning principal  $G$ -bundles over regular semi-local domains containing a finite field: II, Preprint, May 2014.
- [Pop] Dorin Popescu. General Néron desingularization and approximation. *Nagoya Math. J.*, 104:85–115, 1986.
- [Poo] *Poonen, B.*, Bertini theorems over finite fields, *Annals of Mathematics*, 160 (2004), 1099–1127.
- [PPS] I. Panin, V. Petrov, and A. Stavrova. Grothendieck-Serre conjecture for adjoint groups of types  $E_6$  and  $E_7$  and for certain classical groups. *ArXiv e-prints*, 0905.1427, December 2009.
- [PS1] Ivan A. Panin and Andrei A. Suslin. On a conjecture of Grothendieck concerning Azumaya algebras. *St.Petersburg Math. J.*, 9(4):851–858, 1998.
- [PS2] Victor Petrov and Anastasia Stavrova. Grothendieck-Serre conjecture for groups of type  $F_4$  with trivial  $f_3$  invariant. *ArXiv e-prints*, 0911.3132, November 2009.
- [PSV] I. Panin, A. Stavrova, and N. Vavilov. On Grothendieck-Serre’s conjecture concerning principal  $G$ -bundles over reductive group schemes:I. *ArXiv e-prints*, 0905.1418v3, April 2013.
- [Rag1] Madabusi S. Raghunathan. Principal bundles admitting a rational section. *Invent. Math.*, 116(1-3):409–423, 1994.
- [Rag2] Madabusi S. Raghunathan. Erratum: “Principal bundles admitting a rational section” [*Invent. Math.* **116** (1994), no. 1-3, 409–423; MR1253199 (95f:14093)]. *Invent. Math.*, 121(1):223, 1995.
- [Ser] Jean-Pierre Serre. Espaces fibrés algébrique. In *Anneaux de Chow et applications*, *Séminaire Claude Chevalley*, number 3. Paris, 1958.
- [Swa] Richard G. Swan. Néron-Popescu desingularization. In *Algebra and geometry (Taipei, 1995)*, volume 2 of *Lect. Algebra Geom.*, pages 135–192. Int. Press, Cambridge, MA, 1998.
- [Zai] Kirill Zainoulline. On Grothendieck conjecture about principal homogeneous spaces for some classical algebraic groups. *St.Petersburg Math. J.*, 12(1):117–143, 2001.

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